Notes on the Riemann zeta function, 2

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Translator's note

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Introduction

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We denote by $\sum_{\Re\rho>1/2}$ a sum over the possible zeros of $\zeta(s)$ with real part greater than $\frac{1}{2}$, where the zeros of multiplicity m are counted m times. The goal of this note is the proof of the following result.

Theorem. *We have*

$$\frac{1}{2\pi}\int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|\,\mathrm{d}s| = \sum_{\Re\rho>1/2}\log\left|\frac{\rho}{1-\rho}\right|. \tag{1}$$

In particular, the Riemann hypothesis is true if and only if

$$\int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|\,\mathrm{d}s| = 0.$$

Proof. This proof consists of two steps.

First step. We start by stating some properties satisfied by a generic function f in the Hardy space $H^p(\mathbf{D})$, where $\mathbf{D} = \{z\in\mathbb{C} : |z|<1\}$, and p is a positive real number. We denote by f^* the function defined almost everywhere on the trigonometric circle $\partial\mathbf{D} = \{z\in\mathbb{C} : |z|=1\}$ by $f^*(e^{i\theta}) = \lim_{r\to1^-} f(re^{i\theta})$. We use the letter z to denote an element of the trigonometric disc $\mathbf{D}$, and write

$$s = s(z) = \frac{1}{2} + \frac{1+z}{2(1-z)} = \frac{1}{1-z}.$$

This formula defines a conformal representation of the disc $\mathbf{D}$ in the semi-plane $\Re(s) > 1/2$. *p. 285*

By Jensen's formula (see, for example, [4, Theorem 3.61]), we have, for $f(0) \neq 0$ and $r<1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \sum_{\substack{|\alpha|<r\\ f(\alpha)=0}}\log\frac{r}{|\alpha|} \tag{2}$$

where, in the sum, the zeros of multiplicity m are counted m times. Denote by

$$\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,\mathrm{d}\mu(\theta)\right\}$$

the singular interior factor of f. As r tends to 1, Equation (2) becomes (cf. [2])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \sum_{\substack{|\alpha|<1\\ f(\alpha)=0}}\log\frac{1}{|\alpha|} + \int_{-\pi}^{\pi}\mathrm{d}\mu(\theta). \tag{3}$$

This formula is a consequence of the factorisation theorem for functions in H^p; it is stated in [2] for $p=1$, but also holds for all positive values of p.

Second step. Now consider the function

$$f(z) = (s-1)\zeta(s)$$

(where $s = 1/(1-z)$). The elementary properties of the Riemann ζ function (see, for example, [5]) allow us to show that, on one hand, f belongs to the Hardy space $H^{1/3}(\mathbf{D})$, and, on the other hand, that the measure μ associated to the singular interior factor of f is zero (for this latter point, it suffices to reuse the argument developed by Bercovici and Foias for the interior factor of the functions $(\theta-\theta^s)\zeta(s)(s+1/2)/s$, found in the proof of [1, Proposition 2.1]). We can equally show that

$$\int_{-\pi}^{\pi}\log|f^*(e^{i\theta})|\,\mathrm{d}\theta = \int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|\,\mathrm{d}s|,$$

$$\log|f(0)| = 0,$$

$$\sum_{\substack{|\alpha|<1\\ f(\alpha)=0}}\log\frac{1}{|\alpha|} = \sum_{\Re\rho>1/2}\log\left|\frac{\rho}{1-\rho}\right|.$$

With all this information, our result follows from Equation (3). □

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We finish with some remarks. There are statements related to ours in the works [6,7] of Wang and Volchkov. It is even possible that Jensen himself was aware of Equation (1) (the reader can consult the article [3] where Jensen informs Mittag–Leffler of his discovery of Equation (2)). It seem interesting, however, to present things as we have done here, and this is for the following three reasons:

a. Equation (1) is simpler than those that appear in [6,7];
b. we show here that, to establish Equation (1), it is natural to place ourselves in the framework of Hardy spaces;
c. the form of the integral in Equation (1) allows us to interpret this result via Brownian motion, as we show below.

Denote by $Z = X + iY$ the planar Brownian motion from 0 (or from 1), and by $Z_{T_{1/2}} = \frac{1}{2} + iY_{T_{1/2}}$ its first point of impact on the critical line $\Re s = 1/2$, where $T_{1/2} := \inf\{t : X_t = 1/2\}$. We know that $Y_{T_{1/2}}$ follows a Cauchy law with parameter 1/2. In other words, the law of $Y_{T_{1/2}}$ has density $1/2\pi(1/4 + t^2)$. Thus the second part of the theorem can be stated in the following manner: the Riemann hypothesis is true if and only if

$$\mathbb{E}[\log|\zeta(Z_{T_{1/2}})] = 0.$$

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