

Double categories and structured categories

Charles Ehresmann

Note from the translator. *This document is a translation from French of the article*

EHRESMANN, CHARLES. “Catégories doubles et catégories structurées”. *Comptes rendus de l’Académie des Sciences* **256** (1963), 1198–1201. Presented by Jean Leray. 4th of February, 1963.

produced with kind permission from Andrée Ehresmann and Comptes rendus de l’Académie des Sciences.

This version has also incorporated comments and errata from [OC, Comments on Part III-1, p. 339]. Some of these correct minor typos; some are supplementary comments by Andrée Ehresmann and have been included verbatim, preceded by [Comm.] and their comment number (when relevant). Since all the footnotes in the original were citations, we have moved these to the bibliography, which means that the footnote numbering no longer agrees with that of the original. The works in the bibliography have been numbered according to their number in [OC]. The page numbers of the original article are included at the relevant locations in the margins of this version.

— Timothy Hosgood (translator)

Abstract

Definition of structured categories; the particular case of double categories, which admit a category of quadruplets as a quotient category.

1 Double categories

[Comm.] *This note is developed in [63].*

| p. 1198

Definition. We define a *double category* to be a class \mathcal{C} endowed with two composition laws, denoted \bullet and \perp , satisfying the following conditions:

1. (\mathcal{C}, \bullet) is a category, denoted \mathcal{C}^\bullet ; the right and left units of $f \in \mathcal{C}$ will be denoted by $\alpha^\bullet(f)$ and $\beta^\bullet(f)$ respectively, and the class of units by \mathcal{C}_0^\bullet ;
2. (\mathcal{C}, \perp) is a category, denoted \mathcal{C}^\perp ; the units of $f \in \mathcal{C}^\perp$ will be denoted by $\alpha^\perp(f)$ and $\beta^\perp(f)$ respectively, and the class of units by \mathcal{C}_0^\perp ;
3. The maps α^\bullet and β^\bullet (resp. α^\perp and β^\perp) are functors from \mathcal{C}^\perp to \mathcal{C}^\bullet (resp. from \mathcal{C}^\bullet to \mathcal{C}^\perp);
4. *Axiom of permutability.* If the composites $k \bullet h$, $g \bullet f$, $k \perp g$, and $h \perp f$ are defined, then

$$(k \bullet h) \perp (g \bullet f) = (k \perp g) \bullet (h \perp f).$$

Let \mathcal{C} be a class endowed with two composition laws \bullet and \perp satisfying axioms 1 and 2; consider the following axioms:

- 3'. \mathcal{C}_0^\bullet (resp. \mathcal{C}_0^\perp) is stable with respect to \perp (resp. to \bullet);
- 4'. If the composites $k \bullet h$, $g \bullet f$, $k \perp g$, and $h \perp f$ are defined, then both $(k \bullet h) \perp (g \bullet f)$ and $(k \perp g) \bullet (h \perp f)$ are defined and are equal to one another.
5. For all $f \in \mathcal{C}$, we have

$$\begin{aligned}\alpha^*(\alpha^\perp(f)) &= \alpha^\perp(\alpha^*(f)), & \beta^*(\beta^\perp(f)) &= \beta^\perp(\beta^*(f)); \\ \alpha^*(\beta^\perp(f)) &= \beta^\perp(\alpha^*(f)), & \alpha^\perp(\beta^*(f)) &= \beta^*(\alpha^\perp(f)).\end{aligned}$$

Proposition. For $(\mathcal{C}, \bullet, \perp)$ to be a double category, it is necessary and sufficient that conditions 1, 2, 3', 4', and 5 be satisfied. In this case, \mathcal{C}_0^\perp (resp. \mathcal{C}_0^\bullet) is a subcategory of \mathcal{C}^\bullet (resp. \mathcal{C}^\perp).

A double subcategory of a double category \mathcal{C} is a subclass \mathcal{C}' of \mathcal{C} that is a subcategory of \mathcal{C}^\bullet and of \mathcal{C}^\perp ; then \mathcal{C}' is a double category for the composition laws induced by \bullet and \perp .

Definition. Let \mathcal{C} be a double category; we define a *left ideal*¹ (resp. *right ideal*) of \mathcal{C}^\perp to be a subcategory I^\perp of \mathcal{C}^\perp such that $\mathcal{C} \bullet I^\perp = I^\perp$ (resp. $I^\perp \bullet \mathcal{C} = I^\perp$), where $\mathcal{C} \bullet I^\perp$ (resp. $I^\perp \bullet \mathcal{C}$) is the class of composites $f \bullet g$ (resp. $g \bullet f$) for $g \in I^\perp$ and $f \in \mathcal{C}$. We similarly define an *ideal* of \mathcal{C}^\bullet .

| p. 1199

Proposition. Let \mathcal{C} be a double category; a left ideal I^\perp of \mathcal{C}^\perp is a species of structures² [47b, 55] over \mathcal{C}^\bullet for the composition law $(f, g) \mapsto f \bullet g$ if and only if $f \bullet g$ is defined, where $f \in \mathcal{C}$ and $g \in I^\perp$. The corresponding category $\mathcal{E}(I^\perp)$ of hypermorphisms [47b, 55] is a double category for the composition laws

$$(f', g') \bullet (f, g) = (f' \bullet f, g)$$

if and only if $g' = f \bullet g$; further

$$(f', g') \perp (f, g) = (f' \perp f, g' \perp g)$$

if and only if $f' \perp f$ and $g' \perp g$ are defined.

2 Double categories of squares

Let \mathcal{C}_1 and \mathcal{C}_2 be two categories with the same class of units. Let $\square(\mathcal{C}_2, \mathcal{C}_1)$ be the set of quadruples (g_2, g_1, f_1, f_2) , with $f_i, g_i \in \mathcal{C}_i$ for $i = 1, 2$, such that

$$\begin{aligned}\alpha(f_1) &= \alpha(f_2), & \alpha(g_1) &= \beta(f_2); \\ \beta(f_1) &= \alpha(g_2), & \beta(g_1) &= \beta(g_2).\end{aligned}$$

We define two composition laws on $\square(\mathcal{C}_2, \mathcal{C}_1)$:

¹[Comm. 1.6] This definition does not agree with the usual one ([73, 122]) in which a left ideal (or sieve) J of \mathcal{C} is a subclass of \mathcal{C} such that $J \bullet \mathcal{C} \subset J$.

²[Comm. 2.1] For the definition of species of structures and hypermorphism categories (introduced in [47a]), cf. [63, § I, 2–3]; also ([Comm. 25.2]) the set-valued functor associated to $(\mathcal{C}^\bullet, \beta, \alpha^{-1}(e))$ is the partial Hom functor $\text{Hom}(e, -): \mathcal{C} \rightarrow \text{Set}$.

- *Longitudinal multiplication*

$$(g'_2, g'_1, f'_1, f'_2) \square (g_2, g_1, f_1, f_2) = (g'_2, g'_1 g_1, f'_1 f_1, f_2)$$

if and only if $f'_2 = g_2$;

- *Lateral multiplication*

$$(g'_2, g'_1, f'_1, f'_2) \boxminus (g_2, g_1, f_1, f_2) = (g'_2 g_2, g'_1, f_1, f'_2 f_2)$$

if and only if $f'_1 = g_1$.

Proposition. $\square(\mathcal{C}_2, \mathcal{C}_1)$ is a double category for longitudinal and lateral multiplication.

Suppose that $\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2$; recall [47b, 55] that a *square* in \mathcal{C} is an element $(g_2, g_1, f_1, f_2) \in \square(\mathcal{C}, \mathcal{C})$ such that $g_1 f_2 = g_2 f_1$.

Corollary. The class $\square \mathcal{C}$ of squares in \mathcal{C} is a double subcategory of $\square(\mathcal{C}, \mathcal{C})$.

Theorem. Let \mathcal{C} be a double category; then \mathcal{C}^\bullet admits a subcategory³ of the longitudinal category $\square(\mathcal{C}_0^\bullet, \mathcal{C}_0^\perp)$ as a quotient category [47b, 55], where \mathcal{C}_0^\bullet (resp. \mathcal{C}_0^\perp) is endowed with its structure as a subcategory of \mathcal{C}^\perp (resp. of \mathcal{C}^\bullet).

3 Functors into a double category

Let Γ be a category and \mathcal{C} a double category; let $\mathcal{F}(\mathcal{C}^\bullet, \Gamma)$ be the class of functors from Γ to \mathcal{C}^\bullet .

Proposition. $\mathcal{F}(\mathcal{C}^\bullet, \Gamma)$ is a category for the composition law $(\Phi', \Phi) \mapsto \Phi' \perp \Phi$, where $(\Phi' \perp \Phi)(f) = \Phi'(f) \perp \Phi(f)$, if and only if $\Phi'(f) \perp \Phi(f)$ is defined for all $f \in \mathcal{C}$.

| p. 1200

Definition. Let \mathcal{C} and \mathcal{C}_1 be two double categories; we define a *double functor* from \mathcal{C} to \mathcal{C}_1 to be a map Φ from \mathcal{C} to \mathcal{C}_1 such that Φ is a functor from \mathcal{C}^\bullet to \mathcal{C}_1^\bullet and a functor from \mathcal{C}^\perp to \mathcal{C}_1^\perp . The class of double functors from \mathcal{C} to \mathcal{C}_1 is denoted $\mathcal{F}(\mathcal{C}_1, \mathcal{C})$.

[Comm. 3.1¶] The following proposition is not correct: the class of double functors is not closed under source and target maps.

Proposition. $\mathcal{F}(\mathcal{C}_1, \mathcal{C})$ is a subcategory of $\mathcal{F}(\mathcal{C}_1^\bullet, \mathcal{C}^\bullet)$ and of $\mathcal{F}(\mathcal{C}_1^\perp, \mathcal{C}^\perp)$; endowed with the two induced composition laws, $\mathcal{F}(\mathcal{C}_1, \mathcal{C})$ is a double category.

Proposition. Let \mathcal{C} and \mathcal{C}' be two categories; the longitudinal category $\mathfrak{N}(\mathcal{C}', \mathcal{C})$ of natural transformations [52] between functors from \mathcal{C} to \mathcal{C}' can be identified with the category $\mathcal{F}(\boxminus \mathcal{C}', \mathcal{C})$, by identifying the natural transformation (φ', τ, ϕ) with the functor Φ such that

$$\Phi(f) = (\varphi'(f), \tau(\beta(f)), \tau(\alpha(f)), \phi(f))$$

for all $f \in \mathcal{C}$.

Consequently, if $(\mathcal{C}^\bullet, \mathcal{C}^\perp)$ is a double category, then a functor Φ from a category Γ into \mathcal{C}^\bullet can be considered as a generalised natural transformation from $\alpha^\perp \Phi$ to $\beta^\perp \Phi$. We will see another generalisation of natural transformations (the double category of quintets) in a following publication.

³[Comm. 2.2] cf. [63, Theorem 6].

4 Structured categories

Let \mathfrak{M}_0 be a class of classes such that if it contains X then it also contains all the subsets of X , and if it contains X and X' then it also contains the product $X \times X'$; let \mathfrak{M} be the category of all functions from X to Y , where $X, Y \in \mathfrak{M}_0$. Let $(\mathfrak{M}, p, \mathcal{K}, \mathcal{S})$ be a category of homomorphisms [47b, 55], with \mathcal{S} containing the groupoid of invertible elements of \mathcal{K} ; let \mathcal{K}_0 be the class of units of \mathcal{K} ; we identify $h \in \mathcal{K}$ with $(\beta^{\mathcal{K}}(h), p(h), \alpha^{\mathcal{K}}(h))$.

Definition. We define a *structured category in \mathcal{K}* to be a pair (C^\bullet, s) , where C^\bullet is the structure of a category on $C \in \mathfrak{M}_0$, and $s \in \mathcal{K}_0$ with $p(s) = C$, satisfying the following conditions:⁴

1. There exists $s_0 \in \mathcal{K}_0$ such that

$$\begin{aligned} p(s_0) &= C_0^\bullet \\ (s, i_{C_0^\bullet}, s_0), (s_0, \alpha, s), (s_0, \beta, s) &\in \mathcal{K} \end{aligned}$$

where $i_{C_0^\bullet}$ is the canonical injection from C_0^\bullet into C , and α and β are the source and target maps (respectively) in C^\bullet .

2. There exists a product $s \times s$ in \mathcal{K} such that $p(s \times s) = C \times C$; if K is the subclass of $C \times C$ consisting of composable pairs, then there exists $s' \in \mathcal{K}_0$ such that

$$\begin{aligned} p(s') &= K \\ (s \times s, i_K, s') &\in \mathcal{K}. \end{aligned}$$

3. writing x to denote the map $(g, f) \mapsto g \bullet f$ from K to C , the relation $(s \times s, i_K, s') \in \mathcal{K}$ implies $(s, x, s') \in \mathcal{K}$.

Example. A structured category in $\tilde{\mathcal{T}}$, where $\tilde{\mathcal{T}}$ is the category of topologies, is a topological category [50].

| p. 1201

Theorem. For (C^\bullet, C^\perp) to be a double category, it is necessary and sufficient that (C^\bullet, C^\perp) be a structured category in the category \mathcal{F} of functors from one category to another; in this case, (C^\perp, C^\bullet) is also a structured category in \mathcal{F} (the structure on C^\bullet is C^\perp).

Bibliography

- [47a] C. Ehresmann. “Gattungen von lokalen Strukturen”. *Jahresbericht der Deutsche Mathematiker-Vereinigung* 60 (1957), pp. 49–97. URL: <https://eudml.org/doc/146434>.
- [47b] C. Ehresmann. “Espèces de structures locales”. *Séminaire Ehresmann. Topologie et géométrie différentielle* 3 (1961), pp. 1–24. URL: http://www.numdam.org/item/?id=SE_1960-1962__3__A4_0. (Originally appeared in German as [47a]).

⁴[Comm. 3.5+] Conditions 2 and 3 are not strict enough; they are modified in [63] (and in subsequent papers), where s' is required to be a substructure of the product $s \times s$ on \mathcal{K} (and this led to the formal definition of substructures in [63], refined in [69, 66]). Both notions coincide if there exists a substructure on \mathcal{K} , i.e. if there exists a pullback of (α, β) in \mathcal{K} . Cf. [Comm 55.2], where motivations are also given.

- [50] C. Ehresmann. “Catégories topologiques et catégories différentiables”. *Colloque de Géométrie différentielle globale, Bruxelles, Centre Belge de Recherches Mathématiques* (1959), pp. 137–150.
- [OC] *Charles Ehresmann: œuvres complètes et commentées*. Vol. III-1. Amiens, 1980. URL: <https://mes-ehres.fr/ChEh/oeuvres.htm>.
- [52] C. Ehresmann. “Catégorie des foncteurs types”. *Revista de la Unión Matemática Argentina* 20 (1960), pp. 194–209.
- [55] C. Ehresmann. “Élargissements de catégories”. *Séminaire Ehresmann. Topologie et géométrie différentielle* 3 (1961), pp. 25–73. URL: http://www.numdam.org/item/?id=SE_1960-1962__3__A5_0.
- [63] C. Ehresmann. “Catégories structurées”. *Annales scientifiques de l’École normale supérieure* 80 (1963), pp. 349–426. DOI: [10.24033/asens.1125](https://doi.org/10.24033/asens.1125).
- [66] C. Ehresmann. “Structures quotient”. *Commentarii Mathematici Helvetici* 38 (1963), pp. 219–283. DOI: [10.5169/seals-29443](https://doi.org/10.5169/seals-29443).
- [69] C. Ehresmann. “Sous-structures et catégories ordonnées”. *Fundamenta Mathematicae* 54 (1964), pp. 211–228. URL: <https://eudml.org/doc/213757>.
- [73] C. Ehresmann. “Cohomologie sur une catégorie”. *Comptes rendus de l’Académie des Sciences* 259 (1964), pp. 1683–1686.
- [122] C. Ehresmann. *Catégories et structures*. Dunod, 1965.