On modifications and exceptional analytic sets

Hans Grauert

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Translator's note

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The term "modification" first appeared in a 1951 publication [1] by H. Behnke and K. Stein. The authors used it to refer to a process that allows a given complex space to be *p. 331* modified. If X is a complex space, and $N \subset X$ a low-dimensional analytic set, then N is replaced by another set N' such that the complex structure on $X \setminus N$ can be extended to the entire space $X' = (X \setminus N) \cup N'$. The newly obtained complex space X' is then called a *modification* of X.

As already demonstrated in [1], modifications can be very pathological. The interest therefore turned towards special classes of modifications. In [12], H. Hopf considered so-called "σ-processes" on n-dimensional complex manifolds M. These modifications made it possible to replace any point $x \in M$ with a complex projective space $\mathbb{P}^{n-1}$ of dimension $n-1$. The result is a new singularity-free complex manifold M'. There are more general modifications that modify the manifold M at only one point $x \in M$, but the space thus obtained can then contain singular points, i.e. is just a complex space.

This present work deals with the following question. Let X be a complex space, and $A \subset X$ a complex-analytic set. Then when does there exist a modification Y of X where A is replaced by a point y, and such that $X \setminus A = Y \setminus y$?

If such a Y exists, then A is said to be an *exceptional analytic set* in X, and we say that A can be "*collapsed*" to a point.

In general, such a Y does not exist. If X is a complex space, and $A \subset X$ is a compact connected analytic set, then, from a set-theoretic point of view, A can of course always be replaced by a point y_0. Then $Y = (X \setminus A) \cup y_0$ has a canonical topological structure, $Y \setminus y_0 = X \setminus A$ has a complex structure $\mathfrak{S}$, and the identity $X \setminus A \to X \setminus y_0$ can be extended to a continuous map $\lambda\colon X \to Y$. Then λ maps $X \setminus A$ topologically (and even biholomorphically) to Y, and sends A to y_0. If A can now be collapsed to a point, then $\mathfrak{S}$ can be extended to the entire space Y, and λ becomes a holomorphic map $X \to Y$. *p. 332*

We now give an overview of the present work. In §1 we study the concepts of *pseudo-convexity* and *holomorphic convexity* on complex spaces. The reduction theory of Remmert then leads, in §2, to the first general theorem concerning exceptional analytic sets $A \subset X$. In order to simplify the somewhat strong assumption in this theorem, we consider, in §3, a coherent analytic sheaf $\mathfrak{m}$ of germs of holomorphic functions that vanish on A, so that A is exactly the zero set of $\mathfrak{m}$. Using $\mathfrak{m}$, we then endow A with a normal bundle $N_{\mathfrak{m}}$. The structure of $N_{\mathfrak{m}}$ is then critical: A is exceptional if N is weakly negative. We use the word "negative" here in the sense of Kodaira's definition; our result shows that it can be defined in a purely algebraic way in the world of algebraic geometry. Also in §3, we obtain simple criteria for positive (negative) line bundles and characterising projective algebraic spaces. The well-known theorem of Kodaira (that every Hodge manifold X is projective algebraic) is generalised to the case where X is a normal complex space. Then §4 deals with the complex structure of neighbourhoods of analytic sets $A \subset X$, which can be collapsed to a point. The main result of this section is that the neighbourhoods of (special) exceptional analytic sets $A \subset X$ and $A' \subset X'$ are analytically equivalent if they are equivalent in a formal sense. This means that the complex structure can be "calculated," which makes it possible to solve one of Hirzebruch's problems [11], and to transfer the propositions of Enriques and Kodaira from algebraic geometry to complex analysis.

— It should also be mentioned that, using the main results of §4, we construct a complex space X with the following properties:

1. X is connected, compact, and of dimension 2;
2. X is normal, and has only one non-regular points;
3. there exist two analytically and algebraically independent meromorphic functions on X; and
4. X is not an algebraic variety (neither in the projective sense nor the more general sense of Weil).[1]

In contrast, as is well known, Kodaira and Chow [4] have shown that every compact, 2-dimensional complex manifold with two independent meromorphic functions is projective algebraic.

1 Complex spaces, pseudoconvexivity

1.1 —

Complex spaces are defined as in [10]. We always assume that they are reduced: their local rings contain no nilpotent elements. If X is a complex space, $U = U(x)$ a neighbourhood, $A \subset G \subset \mathbb{C}^n$ an analytic set in a domain G of the space $\mathbb{C}^n$ of the n-dimensional complex numbers, and τ a biholomorphic map $U \to A$, then (U, τ, A) is called a chart in X, and τ a biholomorphic embedding of U in G.

We always denote by $\mathscr{O} = \mathscr{O}(X)$ the sheaf of germs of locally holomorphic functions on X. If $A \subset X$ is an analytic subset, then we denote by $\mathfrak{m} = \mathfrak{m}(A) \subset \mathscr{O}$ the sheaf of germs of locally holomorphic functions that vanish on A. By a theorem of Cartan, $\mathfrak{m}$ is coherent.

[1] Some of the results of the present work were discovered in 1959, and published in [7]. There are, however, some errors in [7]: in Theorem 1, it should, of course, read "[. . .] such that G is strongly pseudoconvex and A is the maximal compact analytic subset of G"; furthermore, the criterion in Theorem 2 is only sufficient (see §3.8); Theorem 3 is only proven in the present work in the case where the normal bundle $N(A)$ is weakly negative. — The author has already presented, several times, previously, the example of the complex space X, and, since then, Hironaka has found more interesting examples of complex spaces of this type.

For every subsheaf $\mathscr{I} \subset \mathscr{O}$, let $\mathscr{I}^k$ be the sheaf consisting of germs $f_x = f_{1x} \cdot \ldots \cdot f_{kx}$, where $f_{1x}, \ldots, f_{kx} \in \mathscr{I}_x$ for $x \in X$, $k = 1, 2, \ldots$.

Now[2] let $x \in X$, $\mathfrak{m} = \mathfrak{m}(x)$, and $d(x) = \dim_{\mathbb{C}} \mathfrak{m}_x/\mathfrak{m}_x^2$. If $\psi \colon X \to \mathbb{C}^n$ is a holomorphic map, then ψ defines, at each point $x \in X$, a homomorphism $\psi_x^* \colon \mathscr{O}_z(\mathbb{C}^n) \to \mathscr{O}_x(X)$. This homomorphism maps the maximal ideal $\mathfrak{m}_z \subset \mathscr{O}_z(\mathbb{C}^n)$ to the maximal ideal $\mathfrak{m}_x \subset \mathscr{O}_x(X)$. If the induced map $\mathfrak{m}_z/\mathfrak{m}_z^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective, then we say that ψ is a *regular map* at x. In the case where X is a complex manifold, we see that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is exactly the covariant tangent space of X. Then ψ is regular at $x \in X$ if and only if the Jacobian matrix of ψ at x has rank equal to $\dim_x X$.

We say that a map $\psi \colon X \to \mathbb{C}^n$ is *biholomorphic* if it is a bijection that is regular at every point $x \in X$.

(1). *Let x be a point of a complex space X. Then there exists a neighbourhood $U = U(x)$ and a chart (U, τ, A) with $A \subset G$ and $\dim G = d(x)$. If (U, τ, A) is any such chart, and $\psi \colon U \to \mathbb{C}^n$ is a regular holomorphic map, then there exists an open neighbourhood $V = V(z)$ of $z = \tau(x)$ in G, and a biholomorphic map $\hat{\psi} \colon V \to \mathbb{C}^n$ such that $\psi|W = \hat{\psi} \circ \tau$ (where $W = \tau^{-1}(V)$).*[3]

Of course, $\psi|W$ is then also biholomorphic.

Proof. To prove (1), we may assume that X is an analytic set in a domain $D \subset \mathbb{C}^m$. Let $\hat{\mathfrak{m}}_x$ be the maximal ideal in $\mathscr{O}_x(\mathbb{C}^m)$, and $\mathfrak{i}_x \subset \mathscr{O}_x(\mathbb{C}^m)$ the ideal of germs of holomorphic functions that vanish on $X \subset D$. Let r be the dimension of the image $\mathscr{F}$ of $\mathfrak{i}_x$ under the natural homomorphism $\lambda \colon \mathfrak{i}_x \to \hat{\mathfrak{m}}_x/\hat{\mathfrak{m}}_x^2$. Clearly $m = r + d(x)$. Let $f_1, \ldots, f_r$ be functions that are holomorphic on a neighbourhood of x, with $f_{\nu x} \in \mathfrak{i}_x$, so that the elements $\lambda(f_{\nu x})$ for $\nu = 1, \ldots, r$ span the complex vector space $\mathscr{F}$. Then the rank of the Jacobian matrix of $(f_1, \ldots, f_r)$ in X is equal to r. Then, in a neighbourhood $W = W(x)$, the following properties apply: *p. 334*

1. The functions $f_1, \ldots, f_r$ are holomorphic on W, and vanish on $X \cap W$;
2. $\hat{G} = \{z \in W \mid f_\nu(z) = 0 \text{ for } \nu = 1, 2 \ldots, r\}$ is a $d(x)$-dimensional analytic subset of W that contains no singularities, and which is mapped to a domain in $\mathbb{C}^{d(x)}$ under some biholomorphic map τ.

Now let $A = \tau(X \cap W)$ and $U' = W \cap X$, and we obtain a chart satisfying the required properties.

To prove the second claim of (1), let (U, τ, A) be a chart with $A \subset G$ and $\dim G = d(x)$. We may assume that $U = A$ and that τ is the identity. Then $\lambda(\mathfrak{i}_x) = 0$, since $r = 0$. If $f_1, \ldots, f_n$ are holomorphic functions on U that define a holomorphic map $\psi \colon U \to \mathbb{C}^n$, and if $\hat{f}_1, \ldots, \hat{f}_n$ are holomorphic continuations in an open neighbourhood of x in G, then the rank of the Jacobian matrix of $(\hat{f}_1, \ldots, \hat{f}_n)$ at x is equal to $d(x) = \dim G$. There thus exists a neighbourhood $W = W(x)$ in which the $\hat{f}_1, \ldots, \hat{f}_n$ are holomorphic and give a biholomorphic map $\psi \colon W \to \mathbb{C}^n$. □

By the definition of a complex space, for every point $x \in X$ there is a non-empty system of charts (U, τ, A) such that $x \in U$. As we will show in this section, it is thus possible

[2] A subscript x always denotes the stalk of the sheaf at the point x. If s is a section, then s_x denotes its value at x. Holomorphic functions and sections in $\mathscr{O}$ are always considered to be the same thing. — If F is a complex-analytic vector bundle, then $\underline{F}$ always denotes the sheaf of germs of locally holomorphic sections in F.

[3] This statement and its proof were communicated to me by A. Andreotti.

to transfer the concept of strictly plurisubharmonic functions to the setting of complex spaces.

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