Abelian varieties

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Translator's note

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1 Algebraic groups

p. 9-01

Definition. An *algebraic group* is a pair (G, φ), where G is an algebraic variety and φ is a morphism from $G \times G$ to G that endows the set of points of G with the structure of a group.

Properties. For every point a of G, the translations l_a and r_a defined by $l_a(x) = \varphi(a, x)$ and $r_a(x) = \varphi(x, a)$ are automorphisms of the algebraic variety structure of G.

Every point x of G is simple (indeed, if y is a simple point of G, then there exists an automorphism of G that sends y to x).

If G is connected, then G is irreducible. The map $\pi \colon G \to G$ that, to any point, associates its inverse under the group law is a morphism (and thus an automorphism) of the algebraic variety structure. The proof of this property uses the fact that a bijective (and thus radicial) morphism from one variety to another is birational if it is unramified ([1, Corollary 2 to Proposition 3, Section II, Chapter VI]).

(There would be no problem with asking for G to be connected as part of the definition of algebraic groups).

Definition. An *abelian variety* is an algebraic group whose variety is connected (and thus irreducible) and complete.

We will show that this implies that the group is commutative.

2 A property of complete varieties

Recall that a variety V is said to be complete if, for every variety T and every closed subset $F \subset T \times V$, the projection from F to T is closed. In the classical case, this property is equivalent to compactness.

2.1 Proposition 0

Proposition 0. *Let V be a complete connected variety, T a connected variety, and f a morphism from $T \times V$ to another variety U. Then, if, for some $t_0 \in T$, $f(t_0, v)$ does not depend on v, then*

$$f(t,v) = \varphi(t) \quad \textit{for all } t, v$$

where φ is a morphism from T to U.

p. 9-02

Proof. If $v_1, v_2 \in V$, then the set P_{v_1,v_2} of $t \in T$ such that $f(t,v_1) = f(t,v_2)$ is closed in T. The set $P = \bigcap P_{v_1,v_2}$ of $t \in T$ for which $f(t,v)$ does not depend on v is thus also closed. We will now show that it is also open. If $t_1 \in P$, then $f(t_1, v) = u_1$ for all v. Let $U \setminus F$ be an affine neighbourhood of u_1, with F closed, so $f^{-1}(F)$ is closed, $G = \mathrm{pr}_T(f^{-1}(F))$ is closed, and $T \setminus G$ is a neighbourhood of t_1. For $t' \in T \setminus G$, f defines a morphism $v \mapsto f(t', v)$ from V, which is complete and connected, to $U \setminus F$, which is affine. This map is necessarily constant. Thus $P \supset T \setminus G$ is a neighbourhood of each of its points, i.e. an open subset.

Since T is connected, if $t_0 \in P$, then $P = T$, which finishes the proof. (To see that φ is a morphism, it suffices to take a point $v_0 \in V$, without worrying about the case where $V = \varnothing$). □

Remark. There is an analogous statement in analytic geometry: let V be a compact connected complex-analytic space, T a connected topological space, and f a continuous map from $T \times V$ to another analytic space U, such that f induces, for all $t \in T$, a holomorphic map f_t from $\{t\} \times V$ to U. Then, if f_t is constant for $t = t_0$, then it is constant for all $t \in T$. In other words, a holomorphic map from V to U that is homotopic to a constant map amongst holomorphic maps is constant. The hypothesis that f_t be holomorphic for all t is essential: there are counter-examples with non-Kähler varieties V.

2.2 Consequences of Proposition 0

Proposition 1. *If V is a complete connected variety, T a connected variety, and G an algebraic group, then every morphism $f : T \times V \to G$ is of the form*

$$f(t,v) = \varphi_1(t) \times \varphi_2(v)$$

where φ_1 and φ_2 are morphisms from T and V (respectively) to G.

Proof. Let $t_0 \in T$. Consider $f(t,v) \cdot f(t_0,v)^{-1}$; this is a morphism from $T \times V$ to G that, for $t = t_0$ and arbitrary v, takes the value e (the identity element in G). We thus have

$$\begin{aligned} f(t,v) \cdot f(t_0,v)^{-1} &= \varphi_1(t) \\ \implies f(t,v) &= \varphi_1(t) \cdot f(t_0,v). \end{aligned}$$

which finishes the proof. □

Remark 1. "By an analogous argument we can show," or "by considering the dual group of G, we deduce" that $f(t,v)$ can also be written in the form $\psi_1(v)\psi_2(t)$. *p. 9-03*

Remark 2. If $\varphi_1(t)\varphi_2(v) = \varphi'_1(t)\varphi'_2(v)$, then $\varphi'_1(t) = \varphi_1(t)\cdot a$ and $\varphi'_2(v) = \varphi_2(v)$, where a is some fixed element of G.

Proposition 2. *Let G be a connected group, and V a complete connected variety; suppose that $e \in V \subset G$. Then V is contained in the centre of G.*

Proof. Consider $f : G\times V \to G$ defined by $f(g,v) = v\cdot g\cdot v^{-1}$. If $g = e$, then f does not depend on v. So $f(g,v) = \varphi(g)$. Setting $v = e$, we find that $\varphi(g) = g$, and so $vgv^{-1} = g$, which proves the proposition. □

In particular:

Theorem 0. *The underlying group of an abelian variety is abelian.*

(For another proof of this result, see the Appendix).

3 Functions with values in an abelian variety

Theorem 1. *Every function f on a non-singular variety U with values in an abelian variety A is a morphism.*

This theorem results from the combination of two lemmas.

Lemma 1. *If f is a function defined on a non-singular variety U with values in an algebraic group G, then the set S of points of U where f is not defined is of pure codimension 1.*

Proof. Let φ be the function from $U\times U$ to G defined by $\varphi(u,u') = f(u)f(u')^{-1}$. Let X be an affine neighbourhood of e in G, and φ_0 the function from $U\times U$ to X that only differs from φ in the definition of its domain; there is no worry that $\varphi(U)$ might not be contained in $G\setminus X$, since, if f is defined at u, then φ is defined at (u,u), and there it takes the value e; more precisely, we will show that the following three properties are equivalent:

a. f is defined at u;
b. φ is defined at (u,u);
c. φ_0 is defined at (u,u).

Firstly, (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) is evident.

We now show that (b) $\Longrightarrow$ (a). If φ is defined at (u,u), let $v \in U$ be such that f is defined at v, and such that φ is defined at (u,v); then, for all u' where f is defined,

$$f(u') = f(u')\cdot f(v)^{-1}\cdot f(v) = \varphi(u',v)\cdot f(v).$$

The function f_0 defined by $f_0(u') = \varphi(u',v) \cdot f(v)$ agrees with f, and is defined at the point u, and so (b) $\Longrightarrow$ (a). *p. 9-04*

This shows that the intersection of with the diagonal of the set H of points of $U \times U$ where φ_0 is not defined is S, or rather the image of S under the diagonal map. But the sets of points at which a numerical function on a normal variety is not defined is of pure codimension 1; this is thus also true if we replace "numerical function" with "function with values in an affine variety." So H is of pure codimension 1. Since $U \times U$ is not singular, the codimension in $U \times U$ of $H \cap \Delta$ is $\leqslant \operatorname{codim} H + \operatorname{codim} \Delta = \dim U + 1$, which shows that every component of $H \cap \Delta$ is of codimension $\leqslant 1$ in Δ. □

Remark. The hypothesis that U be non-singular is essential, both for Lemma 1 and for Theorem 1.

Counter-example. Let U be a cone in K^3 that has a cubic G of genus 1 in the 2-dimensional projective space as its base. Then G can be endowed with a group structure. The projection f from U to G is defined at every point except for the origin; S is thus of codimension 2.

Lemma 2. *If f is a function defined on a normal variety U with values in a complete variety V, then the set S of points of U where f is not defined is of codimension > 1.*

Proof. Since V is complete, there exists a variety W contained in D^r (where D denotes the projective line), a morphism p from W to V, and a function s from V to W such that $p \circ s = I_V$. Then $f = p \circ (s \circ f)$ will be defined whenever $s \circ f$ is defined. But $s \circ f$ can be considered as taking values in D^r, since W is closed in D^r, and will thus be defined whenever the r coordinate functions of f are defined. These functions take values in D, and so, since U is normal, the set of points where they are not defined is of codimension > 1 [1, Corollary to Proposition 2, Section 1, Chapter V]. □

4 Functions defined on a product with values in an abelian variety

Theorem 2. *Let X and Y be irreducible varieties, and f a function defined on $X \times Y$ with values in an abelian variety A (whose group law is written additively). Then f is of the form $f(x,y) = f_1(x) + f_2(y)$, where f_1 and f_2 are functions from X and Y (respectively) to A.*

Remark. This implies that f is defined at the points where f_1 and f_2 are both defined, and at these points only. *p. 9-05*

Proof. Let (x_0, y_0) be a simple point of $X \times Y$. By considering the function g on $X \times Y$, defined by

$$g(x,y) = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0),$$

we can reduce to showing that, if a function g with values in A is zero on

$$X \vee Y = \{x_0\} \times Y \cup X \times \{y_0\},$$

then it is zero on $X \times Y$.

We will successively reduce to the following particular cases:

a. X is a curve;
b. X is a complete non-singular curve, and Y is non-singular.

Reduction to (a). The set of points x_1 of X such that there exists an irreducible curve containing x_0 and x_1, where x_0 is a simple point, is dense in X. But (a) implies that g is zero at every point (x_1, y) of the open set on which it is defined, and thus at every point of a dense set, and thus everywhere.

Reduction to (b). If X is an irreducible curve, then there exists a curve X_1 that is both complete and normal (and thus non-singular), as well as a rational equivalence from X to X_1, defined at x_0. Also, if Y_1 is the open subset of simple points of Y, then Y_1 is birationally equivalent to Y. The function g, defined on $X \times Y$, has a corresponding function g_1, defined on $X_1 \times Y_1$, and it clearly suffices to prove the theorem for the function g_1.

Proof in case (b). The variety $X \times Y$ is a product of two non-singular varieties, and thus itself non-singular, and so g is a morphism, by Theorem 1. Its value does not depend on x for $y = y_0$, and thus also for all y, by Proposition 0, since X is complete. It is zero for $x = x_0$, and thus for all x.

This concludes the proof of the theorem. □

Corollary 1. *Every function f defined on an algebraic group G with values in an abelian variety A is of the form $h + a$, where h is a homomorphism from G to A, and a is a constant.*

Proof. Set $h(x) = f(x) - f(e)$. Then the function $g : G \times G \to A$ defined by $g(x, y) = h(x \cdot y)$ is of the form $g_1(x) + g_2(y)$. We can impose that $g_1(e) = 0$, and then $g_2(e) = 0$. By successively taking $x = e$ and then $y = e$, we find that $g_1 = g_2 = h$. Whence $h(x, y) = h(x) + h(y)$. □

Corollary 2. *Every function f defined on a line D with values in an abelian variety A is constant.* *p. 9-06*

Proof. By Theorem 1, f is everywhere defined, and by Corollary 1 applied to the multiplicative group, we have that $f(xy) = f(x) + f(y) + a$. Setting $x = 0$, we have $f(0) = f(0) + f(y) + a$, whence $f(y) = -a$. □

Appendix: adjoint representations

We can also obtain Theorem 0 from the following proposition:

Proposition 3. *Let G be a connected algebraic group, and let C be the centre of G. Then there exists a linear group $L = \mathrm{GL}(m)$ along with an algebraic homomorphism $f : G \to L$ such that $f^{-1}(e) = C$.*

Proof. Let $\mathfrak{a}$ be the local ring of functions on G defined at the identity element e, and let $\mathscr{J}$ be its maximal ideal; set $T_n = \mathscr{J}/\mathscr{J}^n$. For all n, T_n is a finite-dimensional vector space. Every element $x \in G$ defines an inner automorphism $\alpha(x)\colon G \to G$ that induces an automorphism $\mathrm{Ad}_n(x)\colon T_n \to T_n$. Let C_n be the kernel of $\mathrm{Ad}_n : G \to \mathrm{GL}(T_n)$. The C_n form a decreasing sequence of subvarieties of G. Such a sequence is stationary, and so there exists some n_0 such that $C_n = C_{n_0}$ for all $n \geqslant n_0$. We now show that $C_{n_0} = C$: if $x \in C_{n_0}$, then $\mathrm{Ad}_n(x)$ is the identity for all n, and so the automorphism of $\mathfrak{a}$ defined by the inner automorphism $\alpha(x)$ of G is the identity, since the local ring $\mathfrak{a}$ is separated. Consequently, since G is connected (and thus irreducible), $\alpha(x)$ is the identity. Whence the proposition, taking $L = \mathrm{GL}(T_{n_0})$. □

Remark. In characteristic $p \neq 0$, the monomorphism $G/C \to L$ is not necessarily an isomorphism from G/C to its image. For example, consider $G = k^* \times k$ endowed with the group law $\varphi((a,b),(a',b)) = (aa', b + a^p b')$.

We deduce Proposition 2 and Theorem 0 from Proposition 3 by noting that L is an affine variety, and that, if V is complete and connected, then every morphism from V to L is constant.

Bibliography

[1] C. Chevalley. *Fondements de la Géométrie algébrique*. Paris, Secrétariat mathématique, 1958.